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Separation and Solution of Spin 1 Field Equation and Particle Production in Lemaître-Tolman-Bondi Cosmologies

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1. Introduction

An attractive issue in general relativity is the separation, and possibly the solution, of field equation of arbitrary spin in space-time of physical relevance, especially from the cosmological point of view. The knowledge of the normal mode solutions is a basic tool in view of a quantization of the field that in turns can lead to a further adjustement of the theoretical formulation of the cosmological model.

In case of the Robertson-Walker (RW) space-time metric, that is the base of spherically symmetric homogeneous standard cosmology (Weinberg, 1972), the problem has been widely considered (Penrose and Rindler, 1984; Fulling, 1989; Parker and Toms, 2009). Recently that goal can be found solved, for arbitrary spin value, in RW metric by the Newmann-Penrose formalism (Zecca, 2009). The separation method employed to that end has been developed in the line of Chandrashekar's separation of Dirac equation in Kerr metric (Chandrasekhar, 1983). In the specific case of spin 0, 1/2, 1 it has been pointed out (Zecca, 2009a; 2010a; 2010b) that particle creation (annihilation) in expanding universe is possible. (Particle production by universe expansion was originally discussed by Parker (1969; 1971); see also Parker and Toms, 2009). The presence of this effect modifies the gravitational dynamics of the Universe. An extension of the Standard Cosmology has also been proposed that includes the back reaction due to particle production (Zecca, 2010).

The separation of field equation of arbitrary spin has been obtained also in Schwarzschild metric (Zecca, 2006b). This metric is interesting because it represents the gravitational field outside a spherical central non rotating mass such as stars, planets, black holes, .. . In this metric however the separated radial equation are much more difficult to disentagle.

Another situation of relevance concerns the spherically symmetric non homogeneous metrics, and in particular the one that is the base of the Lemaître-Tolman-Bondi (LTB) cosmological model. This metric represents a spherically symmetric inhomogeneous universe filled with freely falling dust matter without pressure. The model can be completely integrated and the general solution of the Einstein equation depends on three arbitrary functions of the radial coordinate. (For a comprehensive study of the model see Krasinski, 1997). The separation of the field equation for spins 0, 1/2 has been shown to be possible also in this model under a special choice of the mentioned integration functions. The surviving configuration remains

however sufficiently general because the cosmological model still depends on an arbitrary function of the radial coordinate (Zecca, 2000; 2001).
In the line of the above considerations, it would be desirable to extend the solution of the field equation to higher spin values. This seems a difficult task in the LTB metrics. Indeed in curved space-time the spinor formulation of field equation of spin value greater than 1, in general involves the knowledge of the Weyl spinor (e.g., Illge, 1993 and references therein). Contrarily to what happens in the Robertson-Walker (RW) metric, the Weyl spinor does not vanish in the LTB metrics (e. g., Zecca, 2000a) and makes the solution of the field equation much more complex.

Therefore, in the present Chapter, we study the spin 1 field equation in LTB models. This is a case that, as far as the author knows, has not yet been considered. Moreover it is the case of the higher spin values where the field equation is insensible to the presence of the Weyl spinor (Illge, 1993). On physical grounds the interest of the spin 1 field case lies in that in the massless case it can be interpreted, in a standard way, in terms of electromagnetic field and in the massive case in terms of Proca fields (Illge, 1993; Penrose and Rindler, 1984; Zecca, 2006). For what concerns the separation of the equation, it is performed for a general LTB metric by using the Newmann-Penrose formalism based on a previously determined null tetrad frame. At this general level of the metric, the angular dependence separates. The separated angular equations coincide with those relative to spin 1 field in Robertson-Walker and Schwarzschild metric that have been previously integrated (Zecca, 1996; 2005a; 2006b). The complete variable separation can be then achived for a class of LTD cosmological models. This is obtained under a factorization assumption $Y = Z(r)T(t)$ on the time and radial dependence of the physical radius $Y(r,t)$, the same assumption under which the spin 0 and spin 1/2 field equations have been previously separated. There results that the separated radial dependence can be reduced to the solution of two independent disentangled ordinary differential equations. These equations still depend on an arbitrary radial function that is an integration function of the cosmological model. For what concerns the separated time dependence, it can be reduced to the solution of two coupled time equations. These equations do not depend on any arbitrary function and have therefore an absolute character in the class of LTB model satisfying the factorization assumption. In turn the time equations can be decoupled and reduced to ordinary differential equations of known form. However due to the special dependence on the physical parameters, an integration by series, that is explicitly performed in every case, results unavoidable.
Finally a quantization of the scheme is performed by mimicking the procedure previously developed for spin 1 field equation in the RW metric (Zecca, 2009a). In that case, the number of one mode particle production per unit of time at time t was found to be proportional to the Hubble "constant" $\dot{R}(t)/R(t)$. Here the quantization procedure again leads to preview particle creation (annihilation) in expanding universe for the LTB models admitting a factorization assumption of the physical radius Y. Moreover it is coherent with the generally admitted big bang origin assumption of the universe because it avoids considering "in states" with underlying Minkowskian space-time at time $t = -\infty$ as often assumed in different examples (Birrell and Davies, 1982; Moradi, 2008; Parker and Toms, 2009). There results a generalization of the RW case. Here the number of one mode particle creation per unit of time, at a given time, is proportional to $\dot{Y}(r,t)/Y(r,t) = \dot{T}(t)/T(t)$. The quantity of particles produced by universe expansion, does not seem of relevance at a generic time of the cosmological evolution, especially at the present time. Instead, for a cosmological model

admitting a big bang origin, an enormous number of particles is foreseen to be produced near the big bang.

2. Spin 1 field equation in a class of spherically symmetric comoving system.

The spin 1 field equation for particles of mass m_0 can be formulated in a general curved space-time by the spinor equation (Penrose and Rindler, 1984) in terms of the spinors Φ_{AB}, $\Theta_{AX'}$

$$\begin{aligned}\nabla^{A}_{X'}\Phi_{AB} &= -i\mu_*\,\Theta_{BX'}\\ \nabla^{X'}_{A}\Theta_{BX'} &= i\mu_*\,\Phi_{AB}\end{aligned}\tag{1}$$

with $\Phi_{AB} = \Phi_{BA}$, $\sqrt{2}\mu_*$ the mass of the particle, $\nabla_{AX'}$ the covariant spinor derivative. The formulation (1) holds in a general curved space-time (see e.g., Illge, 1993, and references therein). The object is to solve the system of equations (1) in the general comoving spherically symmetric Lemaître-Tolman-Bondi (LTB) metric whose line element is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - e^\Gamma dr^2 - Y^2(d\theta^2 + \sin^2\theta d\varphi^2)\tag{2}$$

with $\Gamma = \Gamma(r,t)$, $Y = Y(r,t)$. (See e.g., Krasinski, 1997). The Newmann-Penrose (1962) formalism is a powerfull tool to that end. Accordingly we consider the null tetrad frame $\{l^i,\, n^i,\, m^i,\, m^{\star i}\}$ that was considered in Zecca 1993, for which the directional derivatives and the non trivial spin coefficients, that we report for reader's convenience, are

$$\begin{aligned}
D &\equiv \partial_{00'} = l^i\partial_i = \frac{1}{\sqrt{2}}(\partial_t + e^{-\Gamma/2}\partial_r),\\
\Delta &\equiv \partial_{11'} = n^i\partial_i = \frac{1}{\sqrt{2}}(\partial_t - e^{-\Gamma/2}\partial_r),\\
\delta &\equiv \partial_{01'} = m^i\partial_i = \frac{1}{Y\sqrt{2}}(\partial_\theta + i\csc\theta\,\partial_\varphi),\\
\delta^\star &\equiv \partial_{10'} = m^{\star i}\partial_i = \frac{1}{Y\sqrt{2}}(\partial_\theta - i\csc\theta\,\partial_\varphi).\\
\rho &= -\frac{1}{Y\sqrt{2}}\left(\dot{Y} + Y'e^{-\Gamma/2}\right),\\
\mu &= \frac{1}{Y\sqrt{2}}\left(\dot{Y} - Y'e^{-\Gamma/2}\right)\\
\beta &= -\alpha = \frac{\cot\theta}{2Y\sqrt{2}},\\
\epsilon &= -\gamma = \frac{\dot{\Gamma}}{4\sqrt{2}}
\end{aligned}\tag{3}$$

where $\dot{Y} = \partial Y/\partial t$, $Y' = \partial Y/\partial r$. For the definitions see e. g., Chandrasekhar, 1983 and Penrose and Rindler, 1984.

By expliciting the covariant spinor derivatives in terms of the directional derivatives and spin coefficients (3) the equation (1) reduces to the system of coupled differential equations

$$\begin{aligned}
&(D-2\rho)\Phi_{10}-(\delta^\star-2\alpha)\Phi_{00}=i\mu_\star\Theta_{00'}\\
&(D-\rho+2\epsilon)\Phi_{11}-\delta^\star\Phi_{10}=i\mu_\star\Theta_{10'}\\
&(\Delta+\mu-2\gamma)\Phi_{00}-\delta\Phi_{01}=-i\mu_\star\Theta_{01'}\\
&(\Delta+2\mu)\Phi_{10}-(\delta+2\beta)\Phi_{11}=-i\mu_\star\Theta_{11'}\\
&(D-\rho)\Theta_{01'}-\delta\Theta_{00'}=-i\mu_\star\Phi_{00}\\
&(D-\rho+2\epsilon)\Theta_{11'}-(\delta+2\beta)\Theta_{10'}+\mu\Theta_{00'}=-i\mu_\star\Phi_{10}\\
&(\delta^\star+2\beta)\Theta_{01'}-(\Delta+\mu-2\gamma)\Theta_{00'}+\rho\Theta_{11'}=-i\mu_\star\Phi_{01}\\
&\delta^\star\Theta_{11'}-(\Delta+\mu)\Theta_{10'}=-i\mu_\star\Phi_{11}
\end{aligned} \tag{4}$$

(Note that the situation is similar to the general case of arbitrary spin field equation in RW space-time (Zecca, 2009) when specialized to spin $s = 1$). To separate the system (4) it is useful to put

$$\begin{aligned}
&\Phi_{AB}(r,\theta,\varphi,t)=\alpha(t)\phi_k(r)S_k(\theta)e^{im\varphi}, \quad k=A+B=0,1,2\\
&\Theta_{00'}(r,\theta,\varphi,t)=A(t)\phi_1(r)S_1(\theta)e^{im\varphi}\\
&\Theta_{10'}(r,\theta,\varphi,t)=A(t)\phi_2(r)S_2(\theta)e^{im\varphi}\\
&\Theta_{01'}(r,\theta,\varphi,t)=-A(t)\phi_0(r)S_0(\theta)e^{im\varphi},\\
&\Theta_{11'}=-\Theta_{00'}
\end{aligned} \tag{5}$$

where, for convenience, we we assume $m = 0, \pm1, \pm2, \ldots$. By using (5) into equation (4) the angular dependence factors out and one is left with the equations in the $r,\ t$ variables

$$\begin{aligned}
&(D-2\rho)(\alpha\phi_1)-\frac{\lambda_1}{Y\sqrt{2}}\alpha\phi_0=i\mu_* A\phi_1\\
&(D-\rho+2\epsilon)(\alpha\phi_2)-\frac{\lambda_2}{Y\sqrt{2}}\alpha\phi_1=i\mu_* A\phi_2\\
&(\Delta+\mu+2\epsilon)(\alpha\phi_0)-\frac{\lambda_3}{Y\sqrt{2}}\alpha\phi_1=i\mu_* A\phi_0\\
&(\Delta+2\mu)(\alpha\phi_1)-\frac{\lambda_4}{Y\sqrt{2}}\alpha\phi_2=i\mu_* A\phi_1\\
&(D-\rho)(A\phi_0)+\frac{\lambda_3}{Y\sqrt{2}}A\phi_1=i\mu_*\alpha\phi_1\\
&(D-\rho+2\epsilon)(A\phi_1)-\mu A\phi_1+\frac{\lambda_4}{Y\sqrt{2}}A\phi_2=i\mu_*\alpha\phi_2\\
&(\Delta+\mu+2\epsilon)(A\phi_1)+\rho A\phi_1+\frac{\lambda_1}{Y\sqrt{2}}A\phi_0=i\mu_*\alpha\phi_1\\
&(\Delta+\mu)(A\phi_2)+\frac{\lambda_2}{Y\sqrt{2}}A\phi_1=i\mu_*\alpha\phi_2
\end{aligned} \tag{6}$$

Instead the angular functions satisfy the equations

$$\begin{aligned} L_1^- S_0 &= \lambda_1 S_1, \\ L_0^- S_1 &= \lambda_2 S_2, \\ L_0^+ S_1 &= \lambda_3 S_0, \\ L_1^+ S_2 &= \lambda_4 S_1, \end{aligned} \tag{7}$$

where it has been set $L_n^{\pm} = \partial_\theta \mp m \csc\theta + n \cot\theta$. λ_i $(i = 1, 2, 3, 4)$ are the corresponding separation constants. These equations are the same of those relative to the separation of spin 1 field in RW space-time (cfr. Zecca 2005; 2009). By setting $\lambda_1\lambda_3 = \lambda_2\lambda_4 = -\lambda^2$ the angular equations can be reduced to an eigenvalue problem (Zecca, 1996) whose solutions are expressible (Zecca, 2005) in terms of Legendre functions and Jacobi polynomials (For the definitions see e.g., Abramovitz and Stegun, 1970):

$$\begin{aligned} &S_{1lm} = (1-\xi^2)^{\frac{m}{2}} P_l^m(\xi), \quad l = |m|, |m|+1, .. \\ &S_{2lm} = (1-\xi)^{\frac{m-1}{2}} (1+\xi)^{\frac{m+1}{2}} P_{l-m}^{(m+1,m-1)}(\xi), \qquad m \geq 1,\ l = m, m+1, .. \\ &S_{2lm} = (1+\xi)^{\frac{|m|-1}{2}} (1-\xi)^{\frac{|m|+1}{2}} P_{l-m}^{(|m|-1,|m|+1)}(\xi),\ \ m \leq 1,\ l = |m|, |m|+1, .. \\ &S_{2l0} = \sin\theta\, P_{l+2}^{(1,1)}(\cos\theta), \quad l = 0, 1, 2, .. \\ &S_{0lm}(\theta) = S_{2l-m}(\theta), \qquad (\xi = \cos\theta), \end{aligned} \tag{8}$$

with λ that takes the values $\lambda^2 = l(l+1)$, $l = 0, 1, 2, ..$ By possibly considering a normalization factor, the angular functions satisfy

$$\int d\Omega\ S_{ilm}(\theta) e^{im\varphi} \left(S_{il'm'}(\theta) e^{im'\varphi} \right)^* = \delta_{ll'}\delta_{mm'} \quad (i = 0, 1, 2) \tag{9}$$

a relation usefull in view of an ortho-normalization of the complete solution of (1).

For what concerns the separation of the r and t dependence in (6), it does not seem to be obtainable in general even by using the explicit expression of the spin coefficients. In the following we confine within a class of LTB model for which Γ is related to the function Y and Y itself can be given in an explicit parametric factorized form.

3. Variable separation in Lemaître - Tolman - Bondi cosmological models.

The system (6) can be further separated in its r, t dependence in a sufficiently large class of cosmological models. Suppose to that end that the universe is filled with freely falling dust like matter without pressure, as seen in the comoving spherically symmetric space-time coordinates (2). If the proper energy momentum tensor is considered, the corresponding Einstein equation can be integrated exactly in parametric form and gives rise to what is widely known as the Lemaître-Tolman-Bondi (LTB) cosmological model. (For a comprehensive study of the model see Krasinski, 1997; in the Newman-Penrose formalism see e.g., Zecca, 1993).

The explicit solution is the following (Demianski and Lasota, 1973)

$$
\begin{aligned}
&Y = G\frac{m(r)}{2E(r)}(\cosh\eta - 1); \quad t = t_0(r) + G\frac{m(r)}{(2E(r))^{\frac{3}{2}}}(\sinh\eta - \eta), \quad \eta > 0, \quad E > 0 \\
&Y = G\frac{m(r)}{-2E(r)}(1 - \cos\eta); \quad t = t_0(r) + G\frac{m(r)}{(-2E(r))^{\frac{3}{2}}}(\eta - \sin\eta), \ 0 \le \eta \le 2\pi, \ E < 0 \\
&Y = \left[\frac{3}{2}(2m(r))^{\frac{1}{2}}(t - t_0(r))\right]^{\frac{2}{3}}, \qquad E = 0
\end{aligned}
\tag{10}
$$

$m(r),\ E(r),\ t_0(r)$ are arbitrary integration functions that depend only on the radial coordinate and G the gravitational constant. In particular $m(r)$ can be interpreted as the mass contained in a sphere of radius Y, $m(r) = 4\pi G\int_0^r \sigma(r,t)Y^2(r,t)Y'(r,t)dr$, $\sigma(r,t)$ being the matter density. Moreover Γ and Y are no more independent but

$$
\exp\Gamma \ = \ \frac{Y'^2(r,t)}{1+2E(r)} \tag{11}
$$

a relation usefull for the following purposes.
Suppose now to choose $t_0(r) = 0$ in every case and, in case $E \neq 0$,

$$
G\,m(r) = \ \big(2|E|\big)^{\frac{3}{2}} \tag{12}
$$

With this choices the physical radius in (10) reads

$$
\begin{aligned}
&Y = E^{\frac{1}{2}}(\cosh\eta - 1); \quad t = \sinh\eta - \eta, \quad \eta > 0, \quad E > 0 \\
&Y = |E|^{\frac{1}{2}}(1 - \cos\eta); \quad t = \eta - \sin\eta, \quad 0 \le \eta \le 2\pi, \quad E < 0 \\
&Y = \left(\frac{9}{2}\right)^{\frac{1}{3}} m^{\frac{1}{3}}\, t^{\frac{2}{3}}, \qquad E = 0
\end{aligned}
\tag{13}
$$

These assumptions are sufficient to separate the system (6). Indeed from (13), Y is in every case of the form $Y = Z(r)T(t)$. By using this factorization and relation (11) in the expression of the directional derivatives and spin coefficients, one is able to separate the time dependence from eq. (6). The result is expressed in terms of the coupled time equation

$$
\begin{aligned}
\dot{\alpha}T + 2\dot{T}\alpha - im_0AT = -ik\alpha \\
A\dot{T} + \dot{A}T - im_0\alpha T = ikA
\end{aligned}
\tag{14}
$$

These equations are formally those of the separation of the spin 1 field equation in RW metric. Therefore the solutions $\alpha_k(t),\ A_k(t)$ satisfy the constraint

$$
T^3(t)\left[A_k(t)\alpha^*_{-k}(t) + A^*_{-k}(t)\alpha_k(t)\right] = const \tag{15}
$$

The result follows from Zecca (2006a) after the substitution $R(t) \ \rightarrow \ T(t)$. Also this property is an usefull tool for the normalization of the complete solution of (1).

Instead, for what concerns the radial dependence, one obtains

$$
\begin{aligned}
ik &= \frac{\sqrt{1+2E}}{Z'}\frac{\phi_1'}{\phi_1} + \frac{2}{Z}\sqrt{1+2E} - \frac{\lambda_1}{Z}\frac{\phi_0}{\phi_1} \\
ik &= \frac{\sqrt{1+2E}}{Z'}\frac{\phi_2'}{\phi_2} + \frac{1}{Z}\sqrt{1+2E} - \frac{\lambda_2}{Z}\frac{\phi_1}{\phi_2} \\
-ik &= \frac{\sqrt{1+2E}}{Z'}\frac{\phi_0'}{\phi_0} + \frac{1}{Z}\sqrt{1+2E} + \frac{\lambda_3}{Z}\frac{\phi_1}{\phi_0} \\
-ik &= \frac{\sqrt{1+2E}}{Z'}\frac{\phi_1'}{\phi_1} + \frac{2}{Z}\sqrt{1+2E} + \frac{\lambda_4}{Z}\frac{\phi_2}{\phi_1}
\end{aligned} \tag{16}
$$

k is a separation constant, the same in all equations, to ensure consistency in the separation procedure.

4. Decoupling and properties of the radial solutions.

The equations (16) are similar to the corresponding ones of the RW metric (Zecca, 2005) and can therefore be disentangled in a similar way. By defining the operator

$$
A_b = \sqrt{1+2E}\Big(\frac{1}{Z'}\frac{d}{dr} + \frac{b}{Z}\Big) - ik, \quad b \in \mathbf{C} \tag{17}
$$

eqs. (16) reads

$$
\begin{aligned}
A_2\phi_1 &= \frac{\lambda_1}{Z}\phi_0 \qquad & A_1\phi_2 &= \frac{\lambda_2}{Z}\phi_1 \\
A_1^*\phi_0 &= -\frac{\lambda_3}{Z}\phi_1 \qquad & A_2^*\phi_1 &= -\frac{\lambda_4}{Z}\phi_2
\end{aligned} \tag{18}
$$

and can be easily reduced to equations in a single function

$$
\begin{aligned}
ZA_2ZA_1^*\phi_0 &= -\lambda_1\lambda_3\phi_0 \\
ZA_1ZA_2^*\phi_1 &= -\lambda_2\lambda_4\phi_1 \\
ZA_1^*ZA_2\phi_1 &= -\lambda_1\lambda_3\phi_1 \\
ZA_2^*ZA_1\phi_2 &= -\lambda_2\lambda_4\phi_2
\end{aligned} \tag{19}
$$

By taking into account that $\lambda_1\lambda_3 = \lambda_2\lambda_4 = -\lambda^2$, one has further that the radial solutions satisfy $\phi_1 \equiv \phi_1^*,\ \phi_0 \equiv \phi_2^*$. Therefore it suffices to solve two independent ordinary differential equations. By expliciting the equations for $\phi_0,\ \phi_1$ one obtains respectively

$$
\begin{aligned}
&\frac{Z}{Z'^2}(1+2E)\phi_0'' + \Big[(1+2E)\Big(\frac{4}{Z'} - \frac{ZZ''}{Z'^3}\Big) + \frac{E'Z}{Z'}\Big]\phi_0' + \\
&\qquad + \Big[\frac{E'}{Z'} + \frac{2-\lambda^2+4E}{Z} + k^2Z + 2ik\sqrt{1+2E}\Big]\phi_0 = 0
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&\frac{Z}{Z'^2}(1+2E)\phi_1'' + \Big[(1+2E)\Big(\frac{4}{Z'} - \frac{ZZ''}{Z'^3}\Big) + \frac{E'Z}{Z'}\Big]\phi_1' + \\
&\qquad + \Big[\frac{2E'}{Z'} + k^2Z + \frac{2-\lambda^2+4E}{Z}\Big]\phi_1 = 0
\end{aligned} \tag{21}
$$

Note that the Robertson-Walker metric is a special case of the LTB metric with $Y = rR(t)$, $Z(r) = r$, $2E(r) = -ar^2, (a = 0, \pm 1)$. One can check that with this choice, eqs. (20), (21) become exactly the separated radial equation of spin 1 field in RW metric that were derived in (Zecca, 2005). In RW flat case, normal modes of the field equation, have also been determined (Zecca, 2006a) and a quantization procedure developed leading to the possibility of particle production in expanding universe (Zecca, 2009a). Consequently a simple extension of the Standard Cosmological model has been proposed to include particle production (Zecca, 2010). Instead in the curved cases of the RW metric the eqs. (20), (21) have been solved by reduction to Heun's equation (Zecca, 2009a) without however succeding in determining the normal modes.
In the LTB case, the solution of the radial equations seems quite difficult for a general $E(r)$. In particular this is due to the presence of the square root term in (20). One could try to reduce the equations by expliciting, as assumed in (13), $Z(r) = |E(r)|^{1/2}$ for $E \neq 0$ and $Z(r) = \left(9m(r)/2\right)^{1/3}$ for $E = 0$. However, even with these specifications into the radial equations, the solution does not become easier.

5. Solution of the separated time equations.

In the previous Sections the spin 1 field equation has been separated in the three classes of LTB cosmological models, each of them depending on an arbitrary radial function. The resulting time equations (14) are, contrarily to the radial equations, independent of any model integration function. Therefore it seems usefull to give the explicit solution of the time equations in each case. By setting $B(t) = \alpha(t)T^2(t)$, $\gamma(t) = A(t)T(t)$ the equations (14) can be easily reported to the form

$$\begin{aligned} &im_0 B = \dot{\gamma}T - ik\gamma, \\ &\ddot{\gamma}T + \dot{\gamma}\dot{T} + \gamma\Big(m_0^2 T + \frac{k^2}{T}\Big) = 0 \end{aligned} \tag{22}$$

In this way it suffices to solve the equation for $\gamma(t)$ to obtain $\alpha(t)$ and $A(t)$. The object is now of integrating the equation (22) for γ by distinguishing according to the different situations of E in (13).

5.1 Time equation for $E = 0$.

Here $T(t) = t^{2/3}$. When substituted into the equation for γ in (22) and then by setting $s = t^{1/3}$ one obtains

$$\frac{d\gamma}{ds^2} + (9m_0^2 s^4 + 9k^2)\gamma = 0 \tag{23}$$

The solution of (23) can be given by both odd and even regular functions that can be determined by series. By setting $\gamma = \sum_0^\infty c_n s^n$ into (23) one has the recurrence relation

$$\begin{aligned} &(n+1)(n+2)c_{n+2} + 9k^2 c_n = 0, \quad n = 0,1,2,3, \\ &(n+1)(n+2)c_{n+2} + 9k^2 c_n + 9m_0^2 c_{n-4} = 0, \quad n = 4,5,\ldots \end{aligned} \tag{24}$$

Two independent integral $\gamma_0,\ \gamma_1$ can be obtaind by setting respectively $c_1 \neq 0,\ c_0 = 0$ and $c_0 \neq 0,\ c_1 = 0$. As a consequence of the recurrence relation (24), the general solution is of the form $\gamma(s) = a_0\gamma_0(s) + a_1\gamma_1(s)$, γ_0, γ_1 being respectively an odd and an even function.

The radius of convergence of the series is different from 0, on account of general results (e.g., Moon and Spancer, 1961; Magnus and Winkler, 1979). One has therefore the t dependence

$$\begin{aligned}\gamma_0(t) &= c_1 t^{\frac{1}{3}} + c_3 t + c_5 t^{\frac{5}{3}} + c_7 t^{\frac{7}{3}} + \dots \\ A_0(t) &= \gamma_0 T^{-1} = \frac{c_1}{t^{\frac{1}{3}}} + c_3 t^{\frac{1}{3}} + c_5 t + c_7 t^{\frac{5}{3}} + \dots\end{aligned} \tag{25}$$

and $\alpha_0(t) = B_0(t)T^{-2}(t)$ where $B_0(t)$ follows from (25), the first equation (22) and the expression of $T(t)$. Similarly for $\alpha_1(t)$, $A_1(t)$.

5.2 Time equation for $E < 0$.

Since in the present case $T(\eta) = 1 - \cos\eta$, $t = \eta - \sin\eta$, the eq. (22) can be reported to a differential equation in the variable η

$$\begin{aligned}&\frac{d^2\gamma}{d\eta^2} + \left[\nu_0 + \nu_1 \cos\eta + \nu_2 \cos 2\eta\right]\gamma = 0, \qquad 0 \le \eta \le 2\pi \\ &\nu_0 = \frac{3}{2}m_0^2 + k^2, \qquad \nu_1 = -2m_0^2, \qquad \nu_2 = \frac{m_0^2}{2}\end{aligned} \tag{26}$$

Note that, by setting $\chi = \eta/2$, the equation (26) assumes the form of a Wittaker-Hill equation (Magnus and Winkler 1979) of period π;

$$\begin{aligned}&\frac{d^2\gamma}{d\chi^2} + \left[\lambda_0 + 4\bar{m}q\cos(2\chi) + 2q^2\cos(4\chi)\right]\gamma = 0 \\ &\lambda_0 = 4k^2 + 3m_0^2, \qquad q = \pm m_0, \qquad \bar{m} = \pm 2m_0\end{aligned} \tag{27}$$

The interest in this form of the equation lies in that it may have periodic solutions of period π or 2π. However this possibility is prevented in the present case because the parameter $\bar{m} = \pm 2m_0$ is not, as required, an integer number (see e.g., Magnus and Winkler 1979, Theorem 7.9), m_0 being the mass of the particle. Therefore it is convenient to solve directly eq. (26) by series. It appears that a solution of (26) can be an odd or an even function, We consider separately the cases. By setting $\gamma(\eta) = \sum_0^\infty c_{2n}\eta^{2n}$ into the equation for γ in (26), one obtains for the coefficients the recurrence relation

$$(2n+2)(2n+1)c_{2n+2} + \nu_0 c_{2n} + \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!}\left[\nu_1 + \nu_2 2^{2j}\right] c_{2n-2j} = 0, \quad n = 0,1,2,\dots \tag{28}$$

If instead one looks for odd solutions, $\gamma(\eta) = \sum_0^\infty c_{2n+1}\eta^{2n+1}$, one finds from (26) the recurrence relation

$$(2n+3)(2n+2)c_{2n+3} + \nu_0 c_{2n+1} + \sum_{j=0}^{n} \frac{(-1)^j}{(2j)!}\left[\nu_1 + \nu_2 2^{2j}\right] c_{2n+1-2j} = 0, \quad n = 0,1,2,\dots \tag{29}$$

In both cases the coefficients are completely determined by the first one. To obtain $\gamma(t)$ one has to reverte the expression $t = \eta - \sin\eta$ to have $\eta = \eta(t)$ to be substituted in the series expression of the solution.

5.3 Time equation for $E > 0$.

By expressing now the unknown function γ in terms of η with $T(\eta) = \cosh\eta - 1,\ t = \sinh\eta - \eta,\ (\eta > 0)$, the γ-equation in (22) becomes

$$\frac{d^2\gamma}{d\eta^2} + \left[m_0^2(\cosh\eta - 1)^2 + k^2\right]\gamma = 0 \tag{30}$$

that can be put into the form

$$\begin{aligned} &\frac{d^2\gamma}{d\eta^2} + \left[\sigma_0 + \sigma_1\cosh\eta + \sigma_2\cosh 2\eta\right]\gamma = 0 \\ &\sigma_0 = k^2 + \frac{3}{2}m_0^2, \qquad \sigma_1 = -2m_0^2, \qquad \sigma_2 = \frac{m_0^2}{2} \end{aligned} \tag{31}$$

The last equation can be integrated by series by distinguishing again between even and odd solutions. By setting $\gamma_1(\eta) = \sum_0^\infty c_{2n}\eta^{2n}$ into (31) one has the recurrence relation for the coefficients c_n's

$$(2n+2)(2n+1)c_{2n+2} + (\sigma_0 + \sigma_1 + \sigma_2)c_{2n} + \sum_{j=1}^{n}\frac{c_{2n-2j}}{(2j)!}\left(\sigma_1 + \sigma_2 2^{2j}\right) = 0, \quad n = 0, 1, 2, \ldots \tag{32}$$

Instead by setting $\gamma_1(\eta) = \sum_0^\infty c_{2n+1}\eta^{2n+1}$ into (31) one has

$$(2n+3)(2n+2)c_{2n+3} + (\sigma_0 + \sigma_1 + \sigma_2)c_{2n+1} + \sum_{j=1}^{n}\frac{c_{2n+1-2j}}{(2j)!}\left(\sigma_1 + \sigma_2 2^{2j}\right) = 0, \quad n = 0, 1, .. \tag{33}$$

Here the general solution, $\gamma(t) = a_1\gamma_1(t) + a_2\gamma_2(t)$, follows again by expressing $\eta = \eta(t)$ into $\gamma_1(\eta),\ \gamma_2(\eta)$.

5.3.1 Time equation for $E > 0$ and large t

In the present case one can also determine the behaviour of the situation for large t (large η). To that end, by setting $y = \exp\eta$, the equation (30) becomes

$$\frac{d^2\gamma}{dy^2} + \frac{1}{y}\frac{d\gamma}{dy} + \left[\frac{m_0^2}{4} - m_0^2\frac{1}{y} + \frac{k^2 + 3m_0^2/2}{y^2} - \frac{m_0^2}{y^3} + \frac{m_0^2}{4}\frac{1}{y^4}\right]\gamma = 0 \tag{34}$$

that is in a suitable form for the mentioned purpose. By looking for asymptotic solutions of the form

$$\gamma(\eta) = y^\delta\, e^\chi \sum_{n=0}^{\infty}\frac{c_{-n}}{y^n} \tag{35}$$

one finds, by inserting into eq. (34),

$$\chi = \frac{-1 \pm \sqrt{1 - m_0^2}}{2}, \qquad \delta = \pm\frac{m_0^2}{\sqrt{1 - m_0^2}} \tag{36}$$

Therefore by considering the dominant term in (35), one has, for $y \to \infty$

$$\gamma(y) \;\sim\; y^{\pm\frac{m_0^2}{\sqrt{1-m_0^2}}}\; e^{\frac{-1\pm\sqrt{1-m_0^2}}{2}y} \tag{37}$$

that is a decaying behaviour, except for $m_0 = 1$ in which case the approximation is not valid. Note that for large t, $t \sim e^{\eta}/2 = y/2$ so that the behaviour (37) is also the same of that of $\gamma(t)$ for large t.

6. Remarks and comments.

In the previous Sections the spin 1 field equation has been separated in LTB space-times and reduced to ordinary differential equations in one variable. The angular dependence of the wave spinor factors out in a general LTB metric. Due to spherical symmetry it is the same that the corresponding one in Robertson-Walker and Schwarzschild metric. The further separation of the time and radial coordinates has been possible in LTB cosmologies for which the physical radius has the factorised form $Y = Z(r)T(t)$. This assumption still let the LTB comological model depend on an arbitrary function $E(r)$ (or $m(r)$). As a consequence the separated time dependence is essentially unique in the sense that it depends only on the sign of E or on its vanishing. The time equations have been separated and integrated in all cases.
Instead the radial dependence is reported to the solution of two independent ordinary differential equations that explicitly depend on E. The choice $E(r) = 0$, $Z(r) = r$, $T(t) = R(t)$ ($R(t)$ the radius on the universe in the RW metric) reduces the scheme to a special case of the RW space-time. In this case the radial equations can be explicitly solved (Zecca, 2005). Moreover if one considers toghether with (1) also its complex conjugate equation, a scalar product, induced by a conserved current, can be defined between solutions of (1). Correspondingly normal modes can be defined, that are the base for a quantization of the scheme. In turn this implies that particle creation is possible and that the number of one mode created particles per unit time in expanding universe is proportional to $\dot{R}(t)/R(t)$ (Zecca, 2009a). These results, applied to the present LTB scheme with $E = 0$, $R(t) = T(t) = t^{2/3}$, give that the number of one mode created particles per unit time is proportional to $\dot{T}/T = 2/(3t)$.
Suppose now $E \neq 0$. The procedure of the mentioned RW case, can be applied to define a scalar product between solution of (1), as induced by the conserved current (Zecca, 2006a; 2009a). This product finally factorizes in a product of reduced scalar products in a single variable as a consequence of the assumption $Y = Z(r)T(t)$. By taking into account the orthogonality relation (9) for the angular solutions, the relation (15) for the time dependence and by proceeding as in Zecca, 2006a, one is finally left with a one dimensional scalar product for the solutions of the radial equations (20), (21). If the assumptions on $E(r)$ are such that the solutions of (20), (21) result ortho-normal in the reduced scalar product, then one recover a set of normal mode for the solutions of (1). Accordingly, a quantization procedure can be devoloped as in the flat RW case (Zecca, 2009a). On account of the complete analogy of the two schemes, again one obtains the results of Zecca (2009a) with the substitution $R(t) \to T(t)$. Therefore (with the mentioned suitable choice of E) the balance $n(t)$ of one mode created and

annihilated particles per unit of time is

$$n(t) \propto \frac{\dot{T}}{T} = \frac{\sinh\eta}{(\cosh\eta - 1)^2}; \quad t = \sinh\eta - \eta, \quad \eta > 0, \qquad E > 0 \tag{38}$$

$$n(t) \propto \frac{\dot{T}}{T} = \frac{\sin\eta}{(1 - \cos\eta)^2}; \quad t = \eta - \sin\eta, \quad 0 \le \eta \le 2\pi, \quad E < 0 \tag{39}$$

Therefore, for an LTB cosmology for which $\dot{Y} = Z(r)\dot{T}(t) \neq 0$ particle production is non trivial. Note that for these models one has

$$Y \propto Z(r)\ t^{\frac{2}{3}}, \qquad t \to 0 \tag{40}$$

$$n(t) \propto \frac{\dot{T}}{T} \propto \frac{2}{3}\frac{1}{t}, \qquad t \to 0 \tag{41}$$

for both $E > 0$ and $E < 0$. Hence the cosmological model admits a big bang origin at time $t = 0$ and, if particle production is taken for grant, there is, near the big bang origin, an enormous production of particles that does not depend on the sign of E. This is in some way the converse of what happens in the flat RW metric where particle production is possible for different cosmological dynamics, but with a well defined spatial configuration.

We now briefly comment the separation method employed here. The complete separation of (6) has been done under the special condition (12) for which the physical radius results to be factorized in the time and radial dependence. It would be interesting to know whether the mentioned condition is also in some sense necessary to obtain separated time and radial equations. This would throw also light in the separation of scalar and Dirac field equations that can be separated in LTB models under the same condition (Zecca, 2009; 2001). Solutions of (6) not involving Y-factorizations would be as well of interest.

Another point is the problem of the separation of field equations of spin values higher than 1 in LTB models. This is attractive because the explicit recursive structure of (4) is the same that in the Robertson-Walker metric that in turn is a special case of the general recursive structure for field equations of arbitrary spin (Zecca, 2009). However, as mentioned in the introduction, the presence of a non vanishing Weyl spinor as it happens in LTB metric (e. g., Penrose and Rindler, 1984; Zecca, 2000a) requires a more complex formulation of the field equation for spin greater than 1 (see e. g., Illge, 1993 and references therein). Also in this case it would be interesting to know whether the condition (11) still plays a central role for the separation of the equation, at least in the simplest case of spin $s = 3/2$. The problem is currently under investigation.

As final comment, if particle production is taken for grant, its effect is of modifying the gravitational dynamics of the universe. Therefore it should be taken into account in the formulation of a cosmological model. A precise formulation of the gravitational modification seems problematic. The previous quantization scheme does indeed foresee particle production but it does not specify where and with what density the particles are produced. However, by mediating over possible spatial distributions, a simple modification of the Standar Cosmological model has been proposed by an ansatz on the definition of energy density and of the pressure of the universe (Zecca, 2010).

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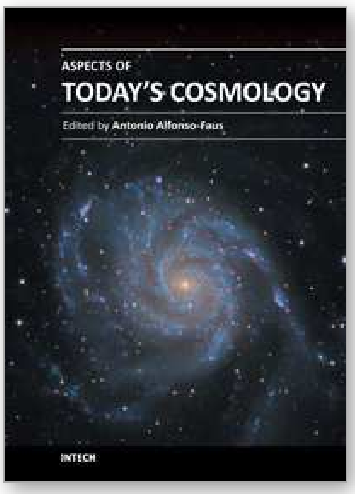

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This book presents some aspects of the cosmological scientific odyssey that started last century. The chapters vary with different particular works, giving a versatile picture. It is the result of the work of many scientists in the field of cosmology, in accordance with their expertise and particular interests. Is a collection of different research papers produced by important scientists in the field of cosmology. A sample of the great deal of efforts made by the scientific community, trying to understand our universe. And it has many challenging subjects, like the possible doomsday to be confirmed by the next decade of experimentation. May be we are now half way in the life of the universe. Many more challenging subjects are not present here: they will be the result of further future work. Among them, we have the possibility of cyclic universes, and the evidence for the existence of a previous universe.

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